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Infinite Divisibility of a Bethe Lattice Ising Model

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It is shown that the probability distribution for the infinite-volume, free-boundary-condition Ising ferromagnet on the Bethe lattice under zero external field is infinitely divisible with respect to the group operation of pointwise multiplication of spin variables.

KEY WORDS: n-point correlations; GHS inequalities; ferromagnet; Markov random field.

1. INTRODUCTION AND PRELIMINARIES

The classical notion of an infinitely divisible probability distribution on the real number line refers to a factorization property with respect to all k-fold convolutions that is known to hold for a great many familiar distributions, e.g., Gaussian, Gamma, Poisson. The general utility of this property in applications lies in the consequent representation of the Fourier transform by the so-called Levy–Khinchine formula (see Feller[(4)]). Previously in the statistical physics literature an interesting perspective on renormalization groups was formulated by Jona-Lasinio[(6)] from the point of view of infinite divisibility. Also, De Coninck and de Gottal[(1,2)] have established an interesting connection with the moment generating function of the distribution of block sums for Ising ferromagnets. These results by De Coninck and de Gottal[(1)] are made especially intriguing by the observation that since the Fourier transform of an infinitely divisible distribution can have no real zeros (see Theorem 2, p. 557, Feller[(4)]), its moment generating function has no zeros on the imaginary axis. Therefore, it follows from the Lee–Yang theorem that the block sums of spins cannot

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have an infinitely divisible distribution in the classical sense. So one expects the connection between such spin distributions and classical infinitely divisible distributions, as in De Coninck and de Gottal,[1] to be somewhat indirect.

In the present paper we pursue a factorization property for the spin distribution of Ising ferromagnets that is nonclassical infinite divisibility only insofar as convolutions are taken with respect to a multiplicative group operation on spin configurations instead of the usual additive group operation on the real number line. This approach also rests on the now standard idea due to Ruelle,[13] Lanford and Ruelle,[8] Dobrushin,[3] and Minlos[9] of defining a (magnetic) Gibbs state as a probability measure on the space Ω of all possible spin configurations. The general utility of these ideas for applications is still based on the corresponding Levy–Khinchine formula, which in this formulation applies to the block correlations $\langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\rangle$. For example, as we will see shortly, certain standard correlation inequalities are made transparent by the Levy–Khinchine formula.

The basic mathematical problem whose solution is sought here requires no special structure for its formulation. Formally, in the case of spins $\{\sigma_n\}$ on a set Λ of sites distributed according to the infinite-volume limit of Ising model probabilites of the form

$$\text{Prob} = \frac{1}{Z}\exp\left\{\beta \sum_{A\subset\Lambda} J_A \prod_{n\in A} \sigma_n\right\} \tag{1.1}$$

the problem is to show that for each integer $k \geqslant 1$ there is a ± 1-valued spin system $\{\mathfrak{S}_n\}$ (defined by some probability distribution on spin configurations depending on k and whose expectation is denoted by $\langle\cdot\rangle_k$) such that for any finite set of sites $D \subset \Lambda$,

$$\left\langle \prod_{n\in D} \sigma_n \right\rangle = \left(\left\langle \prod_{n\in D} \mathfrak{S}_n \right\rangle_k \right)^k, \qquad k = 1, 2,... \tag{1.2}$$

The general extent of the models (1.1) for which the factorization property (1.2) holds is by no means clear. In particular, we do not know whether the standard nearest neighbor models are infinitely divisible at any finite temperature in dimension two or more. In the present paper we consider the simplest exactly soluble case. Namely, we consider the case when $\Lambda \equiv T_N$ consists of the vertices of the infinite connected graph without loops such that each vertex is adjacent to precisely $N+1$ distinct vertices. T_N is referred to as a Bethe lattice or tree graph. The coupling constants will be taken as pairwise and nearest neighbor for this graph structure. Let

$\partial(n)$ denote the (boundary) set of neighbors of the site n. Then in (1.1) we take

$$J_A = \begin{cases} J > 0 & \text{if } A = \{m, n\}, \quad m \in \partial(n) \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

The infinite-volume Ising model on T_N under zero external field is a probability measure on the space $\Omega = \{-1, 1\}^{T_N}$ of spin configurations having the following prescribed local structure in the sense of Lanford, Ruelle, Dobrushin, and Minlos:

$$\begin{aligned} &\text{Prob}(\sigma_n = -1 \mid \sigma_m = -1 \text{ at exactly } r \text{ neighbors } m \text{ of } n) \\ &= \text{const} \cdot e^{\beta J(2r - N - 1)} \\ &= \left[1 + \left(\frac{q}{p}\right)^{2r - N - 1}\right]^{-1}, \qquad r = 0, 1, ..., N+1 \end{aligned} \tag{1.4}$$

where $q = 1 - p$ and

$$\beta J = \tfrac{1}{2} \log(p/q) \tag{1.5}$$

The case in which $p = q = \frac{1}{2}$ corresponds to infinite temperature (or no interaction) and $p \geqslant \frac{1}{2}$ corresponds to ferromagnetism.

Note that in the case $N = 1$, T_1 is simply the one-dimensional integer lattice. Preston[12] has shown that T_N is a *phase transition* graph for the local structure (1.4) if and only if $N \geqslant 2$; see also Müller-Hartmann and Zittartz[10] for an analysis in terms of free energy. However, the probability measure P_p defined by (1.6) below is always among those probabilities having the local structure in (1.4). Moreover, P_p is of Markov chain type with symmetric "transition matrix" M in the sense of Preston[12] and Spitzer.[14]

Let us now turn to the precise formulation of the ideas. For $0 < p < 1$, $q = 1 - p$, let P_p denote the probability distribution given by

$$P_p(X_n = \varepsilon_n, n \in D) = \tfrac{1}{2} p^{n_e(\varepsilon, D)} q^{n_0(\varepsilon, D)} \tag{1.6}$$

where D is a finite connected set of sites in T_N, $\varepsilon = (\varepsilon_n \mid n \in D)$ is a function on D taking values in $\{-1, 1\}$, and

$$n_e(\varepsilon, D) = \sum_{n \in D} \sum_{m \in \partial(n) \cap D} \frac{|\varepsilon_m + \varepsilon_n|}{4} \tag{1.7}$$

$$n_o(\varepsilon, D) = \sum_{n \in D} \sum_{m \in \partial(n) \cap D} \frac{|\varepsilon_m - \varepsilon_n|}{4} \tag{1.8}$$

Under coordinatewise multiplication of configurations and with the product topology, Ω is a compact, Abelian, topological group. The (continuous) characters of Ω are given by

$$\gamma_D(w) = \prod_{n \in D} w_n, \qquad w \in \Omega \tag{1.9}$$

for finite subsets D of T_N. The (topological) dual group of Ω can be identified with the collection $\mathfrak{L}$ of finite subsets of T_N with the discrete topology and with symmetric difference Δ as the multiplication in $\mathfrak{L}$.

A probability measure P on Ω is called *infinitely divisible* if for each integer $k \geqslant 1$, there is a probability measure P_k on Ω such that

$$P = P_k^{*k} \qquad \text{for} \quad k = 1, 2,... \tag{1.10}$$

The following theorem will be established.

Theorem 1.1. For $p \geqslant \frac{1}{2}$, P_p is infinitely divisible.

Theorem 1.1 extends an earlier result of Waymire[15] to a larger class of graphs. The calculations in this latter reference were based on the simple observation that if $\{X_n: n \in Z\}$, $\{Y_n: n \in Z\}$ are independent Markov chains distribution as P_{p_1} and P_{p_2}, respectively, then the conditional distribution of $S_{n+1} = X_{n+1} Y_{n+1}$ given (X_n, Y_n), jointly, is a function of the product $X_n Y_n$. This is no longer true for T_N with $N \geqslant 2$.[5]

As an application of Theorem 1.1, we get formulas for the D-point correlations of P_p for all $D \in \mathfrak{L}$, as given by the following corollary.

Corollary 1.2. Let $p > \frac{1}{2}$. Then for each $D \in \mathfrak{L}$,

$$\left\langle \prod_{n \in D} \sigma_n \right\rangle = \exp\{-2f_p(D)\}$$

where $f_p(D) = +\infty$ for $\# D$ odd, and $f_p(D) = F_p(\gamma_D = -1)$ for $\# D$ even, in which F_p is a σ-finite (nonnegative) measure on Ω, the so-called Levy–Khinchine measure, which is finite outside of every neighborhood of the identity $\sigma_+ \in \Omega$, i.e., $\sigma_+(n) = +1$ for each $n \in T_N$.

Remark. In the case $p = \frac{1}{2}$ the random field has independent values, so that one can write down the formula in Corollary 1.2 by taking $f_{1/2}(\varnothing) = 0$ and $f_{1/2}(D) = +\infty$ otherwise. In general the explicit formula for the Levy–Khinchine measure does not seem to be as simple to describe as in the one-dimensional case. The combinatorics for disconnected blocks are somewhat unwieldy; some calculations are given in Glaffig[5] in this connection. However, as illustrated below, a virtue of the approach here is that some properties of correlations follow from the Levy–Khinchine formula and purely measure-theoretic properties in place of otherwise explicit formulas.

For $p \geqslant \frac{1}{2}$ the so-called *first generalized Griffith's inequality*, as formulated by Kelly and Sherman,[7] is immediately transparent from Corollary 1.2 and the subsequent remark, namely $\langle \prod_{n \in D} \sigma_n \rangle \geqslant 0$. The *second generalized* Griffith's inequality, namely

$$\left\langle \prod_{n \in D} \sigma_n \cdot \prod_{n \in G} \sigma_n \right\rangle \geqslant \left\langle \prod_{n \in D} \sigma_n \right\rangle \left\langle \prod_{n \in G} \sigma_n \right\rangle$$

also follows immediately as a consequence of infinite divisibility through Corollary 1.2 and the simple measure-theoretic properties of F_p. Specifically,

$$\begin{aligned} f_p(D \Delta G) &= F_p(\gamma_{D \Delta G} = -1) \\ &= F_p(\gamma_D = -1, \gamma_G = 1) + F_p(\gamma_D = 1, \gamma_G = -1) \\ &\leqslant F_p(\gamma_D = -1) + F_p(\gamma_G = -1) \\ &= f_p(D) + f_p(G) \end{aligned} \tag{1.11}$$

Note that the GKS inequalities are purely a consequence of infinite divisibility as formulated here. The arguments in (1.11) depend only on the general validity of the Levy–Khinchine formula in Corollary 1.2.

Another, though less standard, set of correlation inequalities, whose feasibility (as well as nonfeasibility) was first considered by Kelly and Sherman,[7] can also be shown to follow from infinite divisibility (see Waymire[15,16]).

2. PROOFS

The objective here is to prove Theorem 1.1 and its corollaries. This will be accomplished in stages.

Lemma 2.1. If $\{Z_n\}$ is a random field on Ω with distribution P, and if for each connected $D \in \mathfrak{L}$,

$$P(Z_n = 1, n \in D) = \tfrac{1}{2} p^{\# D - 1}$$

and

$$P(Z_n = -1, n \in D) = \tfrac{1}{2} p^{\# D - 1}$$

where $0 < p < 1$, then

$$P = P_p$$

Proof. It suffices to show (1.2) for each connected $D \in \mathfrak{L}$. If $\# D = 1$ or 2, then (1.2) follows immediately from simple calculations. For a function $\varepsilon = \{\varepsilon_n: n \in D\}$ with values in $\{-1, 1\}^D$, let

$$D_+ = \{n \in D: \varepsilon_n = +1\}, \qquad D_- = \{n \in D: \varepsilon_n = -1\}$$

For $\# D \geqslant 2$ there is an $m \in D = D_+ \cup D_-$ such that $D \backslash m$ is connected. First consider the case $m \in D_+$. We shall use induction on $\# D_+$ for this case. If $\# D_+ = 1$, then

$$\begin{aligned} P(Z_m = 1, Z_n = -1, n \in D_-) &\\ = P(Z_n = -1, n \in D_-) - P(Z_n = -1, n \in D_- \cup \{m\}) &\\ = \tfrac{1}{2} p^{\# D_- - 1} - \tfrac{1}{2} p^{\# D - 1} &\\ = \tfrac{1}{2} p^{\# D - 2} q &\\ = \tfrac{1}{2} p^{n_e(\varepsilon, D)} q^{n_o(\varepsilon, D)} & \end{aligned}$$

Applying the induction hypothesis, we get

$$\begin{aligned} P(Z_m = 1, Z_n = 1, n \in D_+ \backslash m, Z_n = -1, n \in D_-) &\\ = P(Z_n = 1, n \in D_+ \backslash m, Z_n = -1, n \in D_-) &\\ - P(Z_n = 1, n \in D_+ \backslash m, Z_n = -1, n \in D_- \cup m) &\\ = \tfrac{1}{2} p^{n_e(\varepsilon, D \backslash m)} q^{n_o(\varepsilon, D \backslash m)} - \tfrac{1}{2} p^{n_e(\varepsilon^{(m)}, D)} q^{n_o(\varepsilon^{(m)}, D)} & \end{aligned}$$

where $\varepsilon_n^{(m)} = \varepsilon_n$ if $n \neq m$ and $\varepsilon_n^{(m)} = -\varepsilon_n$ if $n = m$. Formula (1.2) now follows by considering cases and noting parity balance,

$$n_e(\varepsilon, D \backslash m) + n_o(\varepsilon, D \backslash m) = n_e(\varepsilon, D) + n_o(\varepsilon, D) - 1$$

$$n_e(\varepsilon^{(m)}, D) + n_0(\varepsilon^{(m)}, D) = n_e(\varepsilon, D) + n_o(\varepsilon, D)$$

The case $m \in D_-$ can be handled similarly by induction on $\# D_-$. ∎

Because of the asymmetry possibilities, one of the equations in Lemma 2.1 alone is not sufficient to get the result.

Proposition 2.2. $P_{p_1} * P_{p_2} = P_{p_3}$, where

$$p_3 = p_1 p_2 + q_1 q_2, \qquad q_i = 1 - p_i, \qquad i = 1, 2$$

Proof. Let $\{X_n\}$ and $\{Y_n\}$ be independent random fields distributed as P_{p_1} and P_{p_2}, respectively. Then $\{Z_n = X_n Y_n\}$ is distributed as $P_{p_1} * P_{p_2}$. For connected $D \in \mathfrak{L}$,

$$\begin{aligned}
&\text{Prob}(Z_n = 1, n \in D) \\
&= \sum_{\varepsilon = \{\varepsilon_n\} \in \{-1,1\}^D} \text{Prob}(X_n = Y_n = \varepsilon_n, n \in D) \\
&= \sum_{\varepsilon} P_{p_1}(X_n = \varepsilon_n, n \in D)\, P_{p_2}(Y_n = \varepsilon_n, n \in D) \\
&= \sum_{\varepsilon} \frac{1}{4} (p_1 p_2)^{n_e(\varepsilon, D)} (q_1 q_2)^{n_o(\varepsilon, D)} \\
&= \frac{1}{4} \sum_{k=0}^{\# D - 1} 2 \binom{\# D - 1}{k} (p_1 p_2)^k (q_1 q_2)^{\# D - 1 - k} \\
&= \frac{1}{2} (p_1 p_2 + q_1 q_2)^{\# D - 1}
\end{aligned}$$

The case for $\text{Prob}(Z_n = -1, n \in D)$ is similar. ▮

Proof of Theorem 1.1. In view of Proposition 2.2, we have

$$P_p^{*n} = P_{p_n}, \qquad n = 1, 2, \ldots$$

where

$$p_1 = p, \qquad q_1 = q = 1 - p$$

$$p_n = p p_{n-1} + q q_{n-1} = (p - q)\, p_{n-1} + q, \qquad n \geq 2$$

So,

$$p_n = (p - q)^{n-1}(p - \tfrac{1}{2}) + \tfrac{1}{2}, \qquad n = 1, 2, \ldots$$

Now simply observe that $f_n(x) = (2x - 1)^{n-1}(x - \frac{1}{2}) + \frac{1}{2}$, $n > 1$, maps $[\frac{1}{2}, 1]$ onto $[\frac{1}{2}, 1]$ and is strictly increasing. Therefore, P_p is infinitely divisible for $p \geq \frac{1}{2}$. ▮

Note. It follows from the proof of Theorem 1.1 that for $p \geq \frac{1}{2}$, given any positive integer n,

$$P_p = P_{p_n}^{*n} \tag{2.1}$$

where

$$p_n = \tfrac{1}{2} + [2^{-n+1}(p - \tfrac{1}{2})]^{1/n} \tag{2.2}$$

Proof of Theorem 1.2. The result follows immediately from Theorem 1.1 by an application of the Levy–Khinchine representation; see Parthasarathy.[11] ▮

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